Rotations in quantum mechanics[1]

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1 The rotation group

The matrix that rotates a vector (such as momentum) through angle $|\vec{\theta}|$ about an axis in the direction of $\vec{\theta}$ is

$$R(\vec{\theta}) \quad . \tag{1}$$

That is, the rotation transforms a vector $\vec{v}$ to a new vector $\vec{v}_{\rm new}$ with

$$v^i_{\rm new} = \sum_j R(\vec{\theta})_{ij} v^j \quad . \tag{2}$$

The rotations form a *group* in the sense that if $R(\vec{\theta}_1)$ is a rotation matrix and $R(\vec{\theta}_2)$ is a rotation matrix, then the matrix product $R(\vec{\theta}_2)R(\vec{\theta}_1)$ is also a rotation matrix. It is $R(\vec{\theta}_3)$ for some $\vec{\theta}_3$, although it is a little complicated to work out what $\vec{\theta}_3$ is.

The operator that rotates a state through angle $|\vec{\theta}|$ about an axis in the direction of $\vec{\theta}$ is

$$U(R(\vec{\theta})) = e^{-i\vec{\theta}\cdot\vec{J}} \quad . \tag{3}$$

The operators U form a *representation* of the rotation group in the sense that

$$U(R_2 R_1) = U(R_2)U(R_1) \quad . \tag{4}$$

Well,..., this is almost true. Actually for the representations used in quantum mechanics, this holds for rotations that are not too far from the unit matrix. But for large rotations one can also have $U(R_2)U(R_1) = -U(R_2 R_1)$. This happens for systems with total angular momentum 1/2, 3/2, 5/2, ...

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The components of the vector $\vec{J}$ are operators on the space of states. They are the *angular momentum* operators and are the *infinitesimal generators* of rotations. They obey

$$\begin{aligned} [J_x, J_y] &= iJ_z \ , \\ [J_y, J_z] &= iJ_x \ , \\ [J_z, J_x] &= iJ_y \ . \end{aligned} \tag{5}$$

We did not show this in class, but these commutation relations follow directly from the structure of the rotation group. One just has to consider the effect of successive rotations about different axes.

2 Action on a wave function

Suppose that we have a state $|\psi\rangle$ for a spinless particle and we rotate the state. What is the wave function of the new state? It is

$$\langle \vec{x} | e^{-i\vec{\theta}\cdot\vec{J}} | \psi \rangle = \langle R(-\vec{\theta})\vec{x} | \psi \rangle \ , \tag{6}$$

where $\vec{x}' = R(-\vec{\theta})\vec{x}$ is $\vec{x}$ rotated through the opposite angle about the same axis. From this one can derive

$$\vec{J} = \vec{r} \times \vec{p} \ . \tag{7}$$

3 Rotations and spin

That was for a spinless particle. For a particle with spin, we have

$$\langle \vec{x}, i | e^{-i\vec{\theta}\cdot\vec{J}} | \psi \rangle = \sum_j \mathcal{R}(\vec{\theta})_{ij} \langle R(-\vec{\theta})\vec{x}, j | \psi \rangle \ , \tag{8}$$

where i and j are spin indices and the matrices $\mathcal{R}(\vec{\theta})$ form a representation of the rotation group. (For instance, the matrices R that we started with make the spin 1 representation. For spin 1/2 we have other matrices.) This leads to

$$\vec{J} = \vec{r} \times \vec{p} + \vec{S} \ , \tag{9}$$

where $\vec{S}$ is a spin operator with

$$\begin{aligned}
[S_x, S_y] &= iS_z \quad , \\
[S_y, S_z] &= iS_x \quad , \\
[S_z, S_x] &= iS_y \quad .
\end{aligned} \tag{10}$$

just as for $\vec{r} \times \vec{p}$. In a matrix notation, the spin operator acts on the spin indices,

$$\langle \vec{x}, i | S^a | \psi \rangle = \sum_j \mathcal{S}^a_{ij} \langle \vec{x}, j | \psi \rangle \quad . \tag{11}$$