

Some formulas for quantum mechanics¹

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1 Integrals

Fourier transform integral

$$\int_{-\infty}^{+\infty} dx e^{ikx} = 2\pi\delta(k) . \quad (1)$$

2 Vectors

Eigenvectors of the position operator, with conventional normalization

$$\langle \vec{x} | \vec{x}' \rangle = \delta(\vec{x} - \vec{x}') . \quad (2)$$

Completeness relation for these

$$1 = \int d\vec{x} |\vec{x}\rangle \langle \vec{x}| . \quad (3)$$

Eigenvectors of the momentum operator, with conventional normalization

$$\langle \vec{k} | \vec{k}' \rangle = (2\pi)^3 \delta(\vec{k} - \vec{k}') . \quad (4)$$

Completeness relation for these

$$1 = \int \frac{d\vec{k}}{(2\pi)^3} |\vec{k}\rangle \langle \vec{k}| . \quad (5)$$

Momentum eigenstate wave function in position space

$$\langle \vec{x} | \vec{k} \rangle = e^{i\vec{k}\cdot\vec{x}} . \quad (6)$$

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Eigenvectors of an operator with a discrete spectrum, with conventional normalization

$$\langle n|n'\rangle = \delta_{n,n'} . \quad (7)$$

Completeness relation for these

$$1 = \sum_n |n\rangle\langle n| . \quad (8)$$

3 Units

Schrödinger equation for the hydrogen atom (in units with $c = \hbar = 1$)

$$\left[-\frac{1}{2m_e} \nabla^2 - \frac{e^2}{4\pi |\vec{r}|} \right] \psi = E\psi . \quad (9)$$

Here are some important numbers in natural units. First, the fine structure constant

$$\frac{e^2}{4\pi} = \alpha_{\text{em}} \approx \frac{1}{137} . \quad (10)$$

Second, the Bohr radius,

$$a_B = \frac{1}{\alpha_{\text{em}} m_e} \approx 0.0529 \text{ nm} . \quad (11)$$

Third, the hydrogen atom binding energy

$$|E_1| = \frac{\alpha_{\text{em}}^2 m_e}{2} \approx 13.6 \text{ eV} \quad (12)$$

4 Angular momentum

Operator that rotates a state through angle $|\vec{\theta}|$ about axis in the direction of $\vec{\theta}$,

$$e^{-i\vec{\theta}\cdot\vec{J}} . \quad (13)$$

Action on a wave function

$$\langle \vec{x} | e^{-i\vec{\theta}\cdot\vec{J}} | \psi \rangle = \langle R(-\vec{\theta})\vec{x} | \psi \rangle , \quad (14)$$

where $\vec{x}' = R(-\vec{\theta})\vec{x}$ is \vec{x} rotated through the opposite angle about the same axis. From this one can derive

$$\vec{J} = \vec{r} \times \vec{p} . \quad (15)$$

This if for a spinless particle. For a particle with spin, we have

$$\langle \vec{x}, i | e^{-i\vec{\theta} \cdot \vec{J}} | \psi \rangle = \sum_j \mathcal{R}(\vec{\theta})_{ij} \langle R(-\vec{\theta})\vec{x}, j | \psi \rangle , \quad (16)$$

where i and j are spin indices $\mathcal{R}(\vec{\theta})$ is a “representation” of the rotation group. This leads to

$$\vec{J} = \vec{r} \times \vec{p} + \vec{S} , \quad (17)$$

where \vec{S} is a spin operator with

$$\begin{aligned} [S_x, S_y] &= iS_z , \\ [S_y, S_z] &= iS_x , \\ [S_z, S_x] &= iS_y . \end{aligned} \quad (18)$$

just as for $\vec{r} \times \vec{p}$. In a matrix notation, the spin operator acts on the spin indices,

$$\langle \vec{x}, i | S^a | \psi \rangle = \sum_j \mathcal{S}_{ij}^a \langle \vec{x}, j | \psi \rangle . \quad (19)$$

5 Scattering

The differential cross section is defined by

$$\frac{d\sigma}{d\Omega} = \frac{1}{\mathcal{L}} \frac{dN}{d\Omega} , \quad (20)$$

Where dN is the number of particles scattered into a solid angle $d\Omega$ per unit time and \mathcal{L} is the number of incident particles per unit area per unit time. The cross section is related to the scattering amplitude $f(\theta)$ by

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 . \quad (21)$$

Here I use your book's normalization convention for $f(\theta)$. In turn, $f(\theta)$ is defined by the long distance outgoing wave part of a solution of the Schrödinger equation with an incoming plane wave,

$$\psi(r, \theta) \approx e^{ikz} + f(\theta) \frac{e^{ikr}}{r} . \quad (22)$$

(This is just the approximate form for large r .)

The Born approximation for $f(\theta)$ is

$$f(\theta, \phi) = -\frac{m}{2\pi} \int d\vec{r} e^{i(\vec{k}' - \vec{k}) \cdot \vec{r}} V(\vec{r}) . \quad (23)$$

6 Time dependent approach to scattering

Definition of the scattering operator:

$$S = \lim_{\substack{t_I \rightarrow -\infty \\ t_F \rightarrow +\infty}} U(t_F, t_I) , \quad (24)$$

where

$$U(t_F, t_I) = e^{iH_0 t_F} e^{-i(H_0 + V)(t_F - t_I)} e^{-iH_0 t_I} . \quad (25)$$

The formal solution for S is

$$S = T \exp \left(-i \int_{-\infty}^{\infty} d\tau V(\tau) \right) , \quad (26)$$

where T indicates time ordering and

$$V(t) = e^{iH_0 t} V e^{-iH_0 t} . \quad (27)$$

The perturbative expansion of this is

$$\langle p_F | S | p_I \rangle = \langle p_F | p_I \rangle + 2\pi\delta(E_F - E_I) \mathcal{M} , \quad (28)$$

where

$$\mathcal{M} = \sum_{n=1}^{\infty} \langle p_F | (-iV) \frac{i}{E_I - H_0 + i\epsilon} (-iV) \cdots (-iV) | p_I \rangle . \quad (29)$$

Thus we have rules for constructing perturbation theory:

- For each intermediate state with momentum \vec{k}_j , an integration

$$\int \frac{d\vec{k}_j}{(2\pi)^3} . \quad (30)$$

- For each intermediate state with momentum \vec{k}_j , a factor

$$\frac{i}{E_I - E_j + i\epsilon} \quad (31)$$

- For each interaction with the potential, a factor

$$\langle \vec{k}_{j+1} | (-iV) | \vec{k}_j \rangle . \quad (32)$$

Here the first \vec{k}_j is \vec{p}_I and the last is \vec{p}_F . Note that this is ($-i$ times) the Fourier transform of the potential.

The relation of \mathcal{M} to the cross section is

$$d\sigma = \frac{1}{v} \frac{d\vec{p}_F}{(2\pi)^3} |\mathcal{M}|^2 2\pi\delta(E_F - E_I) . \quad (33)$$

The unitarity of the S operator leads to the optical theorem:

$$\sigma_T = -\frac{2}{v_I} \text{Re} \mathcal{M}(p_I, p_I) , \quad (34)$$

7 The EPR paradox and Bell's inequality

Quantum mechanics makes statements about probabilities. However the probabilities in quantum mechanics obey rules that are different from, and inconsistent with, the rules in classical statistics. These differences can be appreciated by studying a system of two spin 1/2 particles with wave function $(|+-\rangle - |-+\rangle)/\sqrt{2}$. One imagines that the two particles are in different places and that different observers A and B measure whether the particle has eigenvalue $+1$ or -1 with respect to the operator $2\vec{s} \cdot \vec{a}$ and $2\vec{s} \cdot \vec{b}$ respectively, where \vec{a} and \vec{b} are unit vectors corresponding to settings of measurement devices of the two observers.

8 Variational principle

The ground state energy of a system has an upper bound,

$$E_0 \leq \langle \Psi | H | \Psi \rangle . \quad (35)$$

This holds for any (normalized) trial wave function $|\Psi\rangle$. To use it, we can choose a set of $|\Psi\rangle$ depending on some parameters and then minimize $\langle \Psi | H | \Psi \rangle$ with respect to the parameters.

9 Adiabatic approximation

If the hamiltonian is a function of time, the eigenstates of $H(t)$ are functions of time $|n(t)\rangle$,

$$H(t)|n(t)\rangle = E_n(t)|n(t)\rangle . \quad (36)$$

As long as the changes in H are slow enough, a system that starts in one eigenstate $|n(0)\rangle$ will stay in that eigenstate

$$|\Psi(t)\rangle \approx e^{i\gamma_n(t)} e^{i\theta_n(t)} |n(t)\rangle . \quad (37)$$

Here $\theta_n(t)$ is the dynamical phase

$$\theta_n(t) = - \int_0^t d\tau E_n(t) . \quad (38)$$

There is an additional phase

$$\gamma_n(t) = \int_0^t d\tau i \langle n(t) | \frac{d}{d\tau} | n(t) \rangle . \quad (39)$$

In the case that H is a function of some parameters R and these parameters depend on t , the additional phase is

$$\gamma_n(t) = i \int_{R_I}^{R_F} \sum_j dR_j \langle n(R) | \frac{\partial}{\partial R_j} | n(R) \rangle . \quad (40)$$

If $R_F = R_I$, we have a line integral around a closed path. In this form, the phase is called Berry's phase.