Physics 417
D. E. Soper
Problems for 17 April, 2006

1. Let $\psi(x)$ be a wave function for a particle in one dimension and let

$$\tilde{\psi}(k) = \int_{-\infty}^{\infty} dx\ e^{-ikx}\,\psi(x) \tag{1}$$

be its Fourier transform. Then the expectation value of the square of the momentum operator is

$$\langle\psi|p^2|\psi\rangle = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\ \tilde{\psi}(k)^* k^2 \tilde{\psi}(k) \ \ . \tag{2}$$

Use the definitions and properties of Fourier transforms to show that this is the same as

$$\langle\psi|p^2|\psi\rangle = \int_{-\infty}^{\infty} dx\ \psi(x)^* \left(\frac{d}{dx}\right)^2 \psi(x) \ \ . \tag{3}$$

2. For a state $|\psi\rangle$ with normalization

$$\langle\psi|\psi\rangle = 1 \ \ , \tag{4}$$

define

$$\begin{aligned} \sigma_x^2 &= \langle\psi|(x-\bar{x})^2|\psi\rangle \ \ , \\ \sigma_p^2 &= \langle\psi|(p-\bar{p})^2|\psi\rangle \ \ . \end{aligned} \tag{5}$$

where

$$\begin{aligned} \bar{x} &= \langle\psi|x|\psi\rangle \ \ , \\ \bar{p} &= \langle\psi|p|\psi\rangle \ \ . \end{aligned} \tag{6}$$

Another useful way to write the definition is

$$\begin{aligned} \sigma_x^2 &= \langle\psi|x^2|\psi\rangle - (\langle\psi|x|\psi\rangle)^2 \ \ , \\ \sigma_p^2 &= \langle\psi|p^2|\psi\rangle - (\langle\psi|p|\psi\rangle)^2 \ \ . \end{aligned} \tag{7}$$

There is a theorem that says that

$$\sigma_x^2 \sigma_p^2 \geq 1/4 \ . \tag{8}$$

(This theorem is proved in your book.) Pick three explicit functions for ψ. For each function, make a graph of the function. Then evaluate σ_x^2, σ_p^2 and their product. I suggest using a gaussian for one function and making a couple of choices that are far from being gaussians.

If you want, you could pick functions for which you can do everything analytically. However, I would suggest picking more wild functions and doing the integrations numerically using Mathematica or your favorite alternative program.

If you want to use a function that has a discontinuous derivative, be careful, since the derivative of a function with a discontinuity does not exist, or in a more sophisticated definition such a derivative has a delta function part.