Choice of units for quantum mechanics[1]

D. E. Soper[2]
University of Oregon
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1 Introduction

People like me who do elementary particle physics usually like their physics formulas to be uncluttered. More specifically, formulas in our book (and most texts for quantum mechanics) contain factors of $\hbar$, c and sometimes ϵ_0 and μ_0. I usually regard these factors as clutter that distracts us from the meaning of the equations and their derivation. These notes concern how to get rid of these factors.

As a first step, consider what the symbol m in $l = 5\,\mathrm{m}$ means. I invite you to think of m as a number. Perhaps it is the number of meters in one Roman league. With that interpretation of unit symbols, an equation like $1\,\mathrm{in} = 0.0254\,\mathrm{m}$ is an sensible equation that says two numbers are equal. Similarly, the s in $t = 4\,\mathrm{s}$ can represent a number.

With that understanding, $c \approx 3.0 \times 10^8\,\mathrm{m/s}$ is a number. If we let s be the number of Roman leagues that light travels in one second, then $c = 1$.

If we adopt the convention that $c = 1$ then we can leave c out of all of our formulas. That saves a lot of writing and it saves us from thinking about where c goes in various formulas.

Still, you may think that we will get into trouble. What if after some calculation we wind up with the answer for a time $t = 2.0\,\mathrm{m}$. Now what? That's easy. We have defined

$$1 = c \approx 3.00 \times 10^8\,\mathrm{m/s} \quad . \tag{1}$$

Therefore $1\,\mathrm{m} \approx [1/(3.0\times 10^8)]\,\mathrm{s}$. Thus $t \approx [2.0/(3.0\times 10^8)]\,\mathrm{s} \approx 6.7\times 10^{-9} s = 6.7\,\mathrm{ns}$. That is, we could measure times in meters, but we want to follow the convention that time should be measured in seconds. To do that, we just use c as a conversion factor according to Eq. (1).

[2]soper@uoregon.edu

2 The speed of light

We define $c = 1$ according to Eq. (1). Then time and distance can be represented in the same units. Also

$$1\,\mathrm{kg} = 1\,\mathrm{kg} \times c^2 \approx 1\,\mathrm{kg} \times (3.0 \times 10^8\,\mathrm{m/s})^2 = 9.0 \times 10^{16}\mathrm{J} \quad . \tag{2}$$

Thus mass and energy can be represented in the same units too.

3 Plank's constant

In the same spirit, we define

$$1 = \hbar \approx 1.05 \times 10^{-34}\,\mathrm{J\,s} \quad . \tag{3}$$

Using this as a conversion factor, energy and inverse time can be represented in the same units. For instance, the relation between the angular frequency $\omega = 2\pi\nu$ of a photon and its energy E is often written as $E = \hbar\omega$. However, we can write this relation simply as $E = \omega$. Then we can convert from joules to inverse seconds using Eq. (3). (However, be careful about the distinction between ω and ν so that you don't drop a 2π.)

The same relation (3) is also

$$1 = \hbar \approx 1.05 \times 10^{-34}\,(\mathrm{kg\,m/s}) \times \mathrm{m} \quad . \tag{4}$$

Using this as a conversion factor, momentum and inverse distance can be represented in the same units.

4 Permittivity of space

In the same spirit, we can define

$$1 = 4\pi\epsilon_0 \approx 1.11 \times 10^{-10}\,\mathrm{C}^2/\mathrm{J\,m} \quad . \tag{5}$$

Then

$$1 = 4\pi\epsilon_0\,\hbar\,c \approx 3.52 \times 10^{-36}\,\mathrm{C}^2 \quad . \tag{6}$$

Thus with this as a conversion factor, we can represent charge with no units.

5 Useful relations

For atomic physics, one usually measures energies in eV, charges in units of e, distances in nm, and times in units of fs. ($1\,\mathrm{nm} = 10^{-9}\,\mathrm{m}$ and $1\,\mathrm{fs} = 10^{-15}\,\mathrm{s}$.) Then the following relations are useful.

First, the proton charge e is dimensionless in the natural units of these notes and e^2 is the fine structure constant

$$e^2 = \alpha_{\rm em} \approx \frac{1}{137} \quad . \tag{7}$$

(I take the symbol e to represent the proton charge, $e > 0$. Sakurai takes e to be the electron charge, $e < 0$. I think that is unfortunate. For instance, I like the unit of energy eV to be positive and to be e times V = 1 Volt.)

Distance scales in atomic physics are set by the electron mass m_e. In particular, $\alpha_{\rm em}$ times m_e gives the inverse of the Bohr radius,

$$a_B = \frac{1}{\alpha_{\rm em}\, m_e} \approx 0.0529\,\mathrm{nm} \quad . \tag{8}$$

Energies in atomic physics are also set by the electron mass. In particular, $\alpha_{\rm em}^2$ times m_e gives twice the hydrogen atom binding energy

$$|E_1| = \frac{\alpha_{\rm em}^2 m_e}{2} \approx 13.6\,\mathrm{eV} \tag{9}$$

To convert between the inverse of energies in eV and times in fs, one can use

$$1 = \hbar \approx 0.658\,\mathrm{eV\,fs} \quad . \tag{10}$$

6 The Schrödinger equation

In our natural units, the Schrödinger equation for the hydrogen atom is

$$\left[-\frac{1}{2m_e}\nabla^2 - \frac{e^2}{|\vec{r}\,|} \right] \psi = E\psi \quad . \tag{11}$$

That's simpler than what is in your book. Note also that with the convention advocated here, we can easily transform to the standard SI units in which distances are measured in meters, times in seconds, masses in kilograms, and charges in coulombs (or currents in amperes).

7 Cautionary note

Actually, $4\pi\epsilon_0 = 1$ is not my favorite convention. For purposes of doing relativistic quantum field theory, I like $\epsilon_0 = 1$. Then the non-relativistic Schrödinger equation for the hydrogen atom is

$$\left[-\frac{1}{2m_e}\nabla^2 - \frac{e^2}{4\pi}\frac{1}{|\vec{r}\,|}\right]\psi = E\psi \quad . \tag{12}$$

This seems ugly because a $1/(4\pi)$ appears. Why is this nice? It's because

$$\int d\vec{r}\; e^{i\vec{q}\cdot\vec{r}}\; \frac{e^2}{4\pi}\frac{1}{|\vec{r}\,|} = \frac{e^2}{\vec{q}^{\,2}} \quad . \tag{13}$$

By having a $1/4\pi$ in coordinate space, we avoid having one in momentum space. However, in order to match the conventions of Sakurai, I use $4\pi\epsilon_0 = 1$ in this course.