Comment on Sakurai problem 1.29(a)[1]

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We would like to show that

$$[\hat{x}_i, G(\hat{\boldsymbol{p}})] = i\frac{\partial G(\hat{\boldsymbol{p}})}{\partial p_i} \quad , \tag{1}$$

where the hats here indicate operators. The right hand side is the partial derivative of $G(\boldsymbol{p})$ with respect to p_i, with the numbers p_1, p_2 and p_3 then replaced by operators $\hat{p}_1$, $\hat{p}_2$ and $\hat{p}_3$. There is a second part, for $[\hat{p}_i, F(\hat{\boldsymbol{x}})]$, but this is really the same problem, just with x and p interchanged. There are two ways to do this problem.

First, let's try it with just operators, using the power series expansion of G,

$$G(\hat{\boldsymbol{p}}) = \sum_{n_1,n_2,n_3} g(n_1, n_2, n_3)\ \hat{p}_1^{n_1}\, \hat{p}_2^{n_2}\, \hat{p}_3^{n_3} \quad . \tag{2}$$

Note the form of this. A typical term might be $g(1, 3, 2)\ \hat{p}_1\, \hat{p}_2^3\, \hat{p}_3^2$. We need to commute this with one of the $\hat{x}$ components. Let's say it is $\hat{x}_1$. Now $\hat{x}_1$ commutes with all of the components $\hat{p}_j$ except for $\hat{p}_1$. We have

$$[\hat{x}_1, \hat{p}_1^{n_i}] = i n_1\, \hat{p}_1^{n_1-1} \quad . \tag{3}$$

(To prove this, you just commute $\hat{x}_1$ past each factor of $\hat{p}_1$. Note that there is a factor $i = \sqrt{-1}$ in addition to the index i.) With this result, we have

$$[\hat{x}_1, G(\hat{\boldsymbol{p}})] = i \sum_{n_1,n_2,n_3} n_1 g(n_1, n_2, n_3)\ \hat{p}_1^{n_1-1}\, \hat{p}_2^{n_2}\, \hat{p}_3^{n_3} \quad . \tag{4}$$

This is the power series expansion of $\partial G(\hat{\boldsymbol{p}})/\partial p_1$. Thus

$$[\hat{x}_1, G(\hat{\boldsymbol{p}})] = i\frac{\partial G(\hat{\boldsymbol{p}})}{\partial p_1} \quad . \tag{5}$$

The same proof holds for $i = 2$ and $i = 3$. Thus we arrive at the result that we wanted.

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Second, we can apply the operator we wanted to a wave function. We consider $\langle \boldsymbol{p} | [\hat{x}_i, G(\hat{\boldsymbol{p}})] | \psi \rangle$ and use the known result for the action of $\hat{x}_i$ on a momentum space wave function, then we use the rule for differentiating a product:

$$
\begin{aligned}
\langle \boldsymbol{p} | [\hat{x}_i, G(\hat{\boldsymbol{p}})] | \psi \rangle &= \langle \boldsymbol{p} | \hat{x}_i G(\hat{\boldsymbol{p}}) - G(\hat{\boldsymbol{p}}) \hat{x}_i | \psi \rangle \\
&= i \frac{\partial}{\partial p_i} \langle \boldsymbol{p} | G(\hat{\boldsymbol{p}}) | \psi \rangle - G(\boldsymbol{p}) \langle \boldsymbol{p} | \hat{x}_i | \psi \rangle \\
&= i \frac{\partial}{\partial p_i} [G(\boldsymbol{p}) \langle \boldsymbol{p} | \psi \rangle] - i G(\boldsymbol{p}) \frac{\partial}{\partial p_i} \langle \boldsymbol{p} | \psi \rangle \\
&= i \frac{\partial G(\boldsymbol{p})}{\partial p_i} \langle \boldsymbol{p} | \psi \rangle \\
&= \langle \boldsymbol{p} | i \frac{\partial G(\hat{\boldsymbol{p}})}{\partial p_i} | \psi \rangle
\end{aligned} \tag{6}
$$

Since this holds for an arbitrary state ψ and an arbitrary $\boldsymbol{p}$, we have the operator statement

$$
[\hat{x}_i, G(\hat{\boldsymbol{p}})] = i \frac{\partial G(\hat{\boldsymbol{p}})}{\partial p_i} \quad , \tag{7}
$$

which is the result that we were looking for.