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Direct numerical integration of one-loop Feynman diagrams for N-photon amplitudes: implementation notes

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Abstract: We present the main methods used for sampling integration points in the calculation of one-loop Feynman diagrams for N-photon amplitudes using the method of direct Monte Carlo integration in momentum space.

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1 Introduction

These are implementation notes to accompany our paper *Direct numerical integration of one-loop Feynman diagrams for N-photon amplitudes* [arXiv: xxxxx]. Specifically, we describe the methods we used for importance sampling in performing the integration by Monte Carlo integration.

We would like to perform an integral of the form

$$I = \int d^4 l \ f(l) \ . \tag{1.1}$$

In a Monte Carlo integration, we choose $N_{\rm pts}$ points $\{\ell_j\}$ at random with a density $\rho(l)$ and evaluate the integrand $f(l)$ at these points. Then the integral is

$$I = \lim_{N_{\rm pts}\to\infty} \frac{1}{N_{\rm pts}} \sum_{j=1}^{N_{\rm pts}} \frac{f(l_j)}{\rho(l_j)} \ \ . \tag{1.2}$$

The integration error with a finite number of points is proportional to $1/\sqrt{N_{\rm pts}}$. The coefficient of $1/\sqrt{N_{\rm points}}$ in the error is smallest if

$$\rho(l) \approx const. \times |f(l)| \ \ . \tag{1.3}$$

That is the ideal, but it is not really possible to achieve this ideal to the degree that one has a one part per mill error with one million points. However, one would certainly like to keep $|f(l)|/\rho(l)$ from being very large. In particular, $f(l)$ is singular along certain lines (the collinear singularities) and at certain points (the soft singularities). We need to arrange that ρ is singular at the same places that f is singular, so that $f(l)/\rho(l)$ is *not* singular anywhere. Since $|f(l)|$ can be very large near the point associated with double parton scattering (for external momentum configurations that are not far enough from a double parton scattering configuration), we also need to arrange that $\rho(l)$ is similarly large near this point.

We construct the desired density in the form

$$\rho(l) = \sum_{J=1}^{N_{\rm alg}} \alpha_J\, \rho_J(l) \quad . \tag{1.4}$$

There are several density functions ρ_J with

$$\int d^4l\ \rho_J(l) = 1 \quad . \tag{1.5}$$

We choose probabilities α_j with

$$\sum_{J=1}^{N_{\rm alg}} \alpha_J = 1 \quad . \tag{1.6}$$

Each ρ_J corresponds to a certain algorithm for choosing a point l. For each new integration point, the computer chooses which algorithm to use with probability α_J.

In the following sections, we describe the more important algorithms used for choosing points l. Some methods are more important than others. On the other hand, there is no cost to keeping a method that is of little importance as long as we set its probability α_J to be small. Thus we have more methods than one would really need.

We apologize that the construction of some of the methods for choosing points is quite *ad hoc*. An interested reader of these notes is advised to write his or her own methods. We supply these notes only to document the code and perhaps to suggest a few useful tricks for importance sampling for this kind of problem.

2 Soft singularity

We consider how to choose points near a soft singularity, that is for $l - Q_i$ small for some choice of i. Let us parametrize l by

$$l - Q_i = x\,(Q_{i+1} - Q_i) + y\,(Q_{i-1} - Q_i) + \ell_\perp \quad , \tag{2.1}$$

where $\ell_\perp \cdot (Q_{i\pm1} - Q_i) = 0$. We then have

$$d^4l = \pi M_i^2\ dx\, dy\, d\vec{\ell}_\perp^{\,2}\, \frac{d\phi}{2\pi} \quad , \tag{2.2}$$

where

$$M_i^2 = |(Q_{i+1} - Q_i) \cdot (Q_{i-1} - Q_i)| \tag{2.3}$$

and where ϕ is the azimuthal angle of $\vec{l}_\perp$ relative to any convenient reference direction.

Let us choose a value of x in the range $-1 < x < 1$ using

$$x = s_x\, r_x^{1/A} \tag{2.4}$$

where $s_x = \pm 1$ gives the sign of x, chosen with probability 1/2 for each sign and where r_x is chosen randomly with a uniform density in $0 < r_x < 1$. Here A is a positive parameter. Then

$$r_x = |x|^A \tag{2.5}$$

and

$$\int_{-1}^{1} dx \cdots = \frac{1}{2} \sum_{s_x} \int_0^1 \frac{2}{A}\, |x|^{1-A}\, dr_x \cdots \quad . \tag{2.6}$$

Similarly, we choose a value of y in the range $-1 < y < 1$ using

$$y = s_y\, r_y^{1/A} \tag{2.7}$$

where $s_y = \pm 1$ and $0 < r_y < 1$. Then

$$\int_{-1}^{1} dy \cdots = \frac{1}{2} \sum_{s_y} \int_0^1 \frac{2}{A}\, |y|^{1-A}\, dr_y \cdots \quad . \tag{2.8}$$

We can generate the azimuthal angle uniformly by choosing r_ϕ uniformly in $0 < r_\phi < 1$ and setting

$$\phi = 2\pi\, r_\phi \quad , \tag{2.9}$$

so that

$$\int_0^{2\pi} \frac{d\phi}{2\pi} \cdots = \int_0^1 dr_\phi \cdots \quad . \tag{2.10}$$

This leaves $\vec{\ell}_\perp^{\,2}$. We first set

$$\vec{\ell}_\perp^{\,2} = |xy| M_i^2 \lambda_\perp \quad , \tag{2.11}$$

where we integrate $\lambda_\perp$ in the range

$$0 < \lambda_\perp < \frac{1}{|xy|} \quad . \tag{2.12}$$

This corresponds to a range

$$0 < \vec{\ell}_\perp^{\,2} < M_i^2 \tag{2.13}$$

for $\ell_\perp^2$. This gives

$$\int_0^{M_i^2} d\vec{\ell}_\perp^{\,2} \cdots = \int_0^{1/(|xy|)} |xy| M_i^2\, d\lambda_\perp \cdots \quad . \tag{2.14}$$

We can choose $\lambda_\perp$ by choosing $r_\perp$ uniformly in $0 < r_\perp < 1$ and setting $\lambda_\perp$ to be the solution of

$$r_\perp = \frac{|xy|^B}{(1+|xy|)^B - |xy|^B}\ \left[(1+\lambda_\perp)^B - 1\right] \ . \tag{2.15}$$

Here B is a positive parameter. The solution is

$$\lambda_\perp = \left(\frac{1}{|xy|^B}\left[(1+|xy|)^B - |xy|^B\right] r_\perp + 1\right)^{1/B} - 1 \ . \tag{2.16}$$

This mapping gives

$$\int_1^{1/(xy)} d\lambda_\perp \cdots = \int_0^1 dr_\perp\, \frac{1}{B}\, |xy|^{-B}\left[(1+|xy|)^B - |xy|^B\right](1+\lambda_\perp)^{1-B}\cdots \ . \tag{2.17}$$

That is

$$\int_0^{M_i^2} d\vec{\ell}_\perp^{\,2}\cdots = \frac{(M_i^2)^B}{B}\int_0^1 dr_\perp\ \left[(1+|xy|)^B - |xy|^B\right](|xy|M_i^2 + \vec{\ell}_\perp^{\,2})^{1-B}\dots \ . \tag{2.18}$$

Combining these results, we have

$$\int d^4l\ \theta(|x|<1)\,\theta(|y|<1)\,\theta(\vec{\ell}_\perp^{\,2} < M_i^2)\cdots = \frac{1}{4}\sum_{s_x,s_y}\int_0^1 dr_x\int_0^1 dr_y\int_0^1 dr_\perp\int_0^1 dr_\phi\ \frac{1}{\rho}\cdots \ , \tag{2.19}$$

where the density of points, ρ, is

$$\begin{aligned}\rho = {} & \frac{A^2B}{4\pi M_i^4}\left[(1+|xy|)^B - |xy|^B\right]^{-1}\frac{1}{|xy|^{1-A}}\ \frac{(M_i^2)^{1-B}}{(|xy|M_i^2 + \vec{\ell}_\perp^{\,2})^{1-B}} \\ & \times\theta(|x|<1)\,\theta(|y|<1)\,\theta(\vec{\ell}_\perp^{\,2} < M_i^2) \ .\end{aligned} \tag{2.20}$$

One might choose $A = B = 1/2$. We could choose smaller values so as to put more points near the singularity at $x = y = 0$ and $\vec{l}_\perp = 0$.

3 Collinear singularity

We will also want to choose points near the collinear singularity along the line $l = Q_i + x(Q_{i+1} - Q_i)$, also taking into account the soft-collinear singularity at $x \to 0$. (For the $x \to 1$ singularity, we can use the same formulas with a little change in notation.)

As before, we parametrize l by

$$l - Q_i = x\,(Q_{i+1} - Q_i) + y\,(Q_{i-1} - Q_i) + \ell_\perp \ , \tag{3.1}$$

where $\ell_\perp \cdot (Q_{i\pm1} - Q_i) = 0$. We then have

$$d^4 l = \pi M_i^2 \; dx\, dy\, d\vec{\ell}_\perp^{\,2}\, \frac{d\phi}{2\pi} \quad , \tag{3.2}$$

where

$$M_i^2 = |(Q_{i+1} - Q_i) \cdot (Q_{i-1} - Q_i)| \tag{3.3}$$

and where ϕ is the azimuthal angle of $\vec{l}_\perp$ relative to any convenient reference direction.

As before, we can generate the azimuthal angle uniformly by choosing r_ϕ uniformly in $0 < r_\phi < 1$ and setting

$$\phi = 2\pi\, r_\phi \quad , \tag{3.4}$$

so that

$$\int_0^{2\pi} \frac{d\phi}{2\pi} \cdots = \int_0^1 dr_\phi \cdots \quad . \tag{3.5}$$

We follow almost the same path as before, choosing a value of x in the range $0 < x < 1$ using

$$x = r_x^{1/A} \quad , \tag{3.6}$$

where r_x is chosen randomly with a uniform density in $0 < r_x < 1$. Here A is a positive parameter. Then

$$r_x = |x|^A \tag{3.7}$$

and

$$\int_0^1 dx \cdots = \int_0^1 \frac{1}{A}\, |x|^{1-A}\, dr_x \cdots \quad . \tag{3.8}$$

Similarly, we choose a value of y using

$$y = s_y\, |y| \tag{3.9}$$

where $s_y = \pm 1$. Then

$$\int_{-\infty}^{\infty} dy \cdots = \frac{1}{2} \sum_{s_y} \int_0^\infty 2\, d|y| \cdots \quad . \tag{3.10}$$

We now let

$$\begin{aligned} \vec{l}_\perp^{\,2} &= M_i^2\, r_z\, r_d^{1/B} \quad , \\ |y| &= \frac{1}{x}\,(1 - r_z)\, r_d^{1/B} \quad . \end{aligned} \tag{3.11}$$

where AB is a positive parameter and we choose r_z and r_d with a uniform distribution in the interval 0 to 1. Then

$$r_d = \left[\left(x|y| M_i^2 + \vec{l}_\perp^{\,2} \right) / M_i^2 \right]^B \quad . \tag{3.12}$$

This gives

$$\int_0^\infty d|y| \int_0^\infty d\vec{l}_\perp^{\,2}\ \theta\big(x|y|M_i^2 + l_\perp^{\,2} < M_i^2\big) \cdots = \frac{M_i^2}{Bx} \int_0^1 dr_z \int_0^1 dr_d\ \left(\frac{x|y|M_i^2 + \vec{l}_\perp^{\,2}}{M_i^2}\right)^{2-B} . \tag{3.13}$$

Combining these results, we have

$$\begin{aligned} \int d^4l\ \theta(0 < x < 1)\, \theta\Big(x|y|M_i^2 + \vec{l}_\perp^{\,2} < M_i^2\Big) \ldots \\ = \frac{1}{2} \sum_{s_y} \int_0^1 dr_x \int_0^1 dr_\phi\ \int_0^1 dr_z \int_0^1 dr_d\ \frac{1}{\rho} \cdots \quad , \end{aligned} \tag{3.14}$$

where the density of points, ρ, is

$$\rho = \frac{AB}{2\pi M_i^4}\ x^A \left(\frac{M_i^2}{x|y|M_i^2 + \vec{l}_\perp^{\,2}}\right)^{2-B} \theta(0 < x < 1)\, \theta\Big(x|y|M_i^2 + \vec{l}_\perp^{\,2} < M_i^2\Big) \quad . \tag{3.15}$$

One might choose $A = B = 1/2$.

4 Two cones, timelike separated

Consider two cones $C_+(Q_i)$ and $C_-(Q_j)$, where

$$(Q_j - Q_i)^2 > 0 \ \ , \quad Q_j^0 - Q_i^0 > 0 \ \ . \tag{4.1}$$

We would like to concentrate points near these two cones, with points especially concentrated near their intersection. Let

$$K = Q_j - Q_i \tag{4.2}$$

and choose a reference frame in which $\vec{K} = 0$ with $K^0 > 0$. Let

$$\ell = l - Q_i \ \ . \tag{4.3}$$

Then the two cones are $\ell^2 = 0$ with $\ell^0 > 0$ and $(\ell - K)^2 = 0$ with $\ell^0 - K^0 < 0$.

We use four variables, ℓ^2, $(\ell - K)^2$, and a unit vector in three dimensions, $\vec{u}$. We note that

$$2\ell \cdot K = \ell^2 - (\ell - K)^2 + K^2 \ \ . \tag{4.4}$$

That is

$$\ell^0 = \frac{\ell^2 - (\ell - K)^2 + K^2}{2K} \ \ . \tag{4.5}$$

Here $K = K^0 = \sqrt{K^2}$. We also note that

$$\vec{\ell}^{\,2} = (\ell^0)^2 - \ell^2 \ \ , \tag{4.6}$$

where ℓ^0 is given in Eq. (4.5). We then define

$$\vec{\ell} = |\vec{\ell}|\,\vec{u} \ . \tag{4.7}$$

This gives us all four components of ℓ.

We can write the result for $\vec{\ell}^{\,2}$, Eq. (4.6), using Eq. (4.5), as

$$4K^2\vec{\ell}^{\,2} = [\ell^2 - (\ell-K)^2]^2 - 2[\ell^2 + (\ell-K)^2]\,K^2 + (K^2)^2 \ . \tag{4.8}$$

Since $\vec{\ell}^{\,2} > 0$, we need

$$2[\ell^2 + (\ell-K)^2]\,K^2 < [\ell^2 - (\ell-K)^2]^2 + (K^2)^2 \ . \tag{4.9}$$

This inequality is satisfied if

$$2[\ell^2 + (\ell-K)^2] < K^2 \ . \tag{4.10}$$

In turn, this is satisfied if the following inequalities hold,

$$\begin{aligned} |\ell^2| &< K^2/4 \ , \\ |(\ell-K)^2| &< K^2/4 \ . \end{aligned} \tag{4.11}$$

Since we want to cover the region of small $|\ell^2|$ and small $|(\ell-K)^2|$, there is no problem with imposing these restrictions on the virtuality ranges to be included.

To change variables to our new set, we write,

$$\begin{aligned} d^4\ell &= d\ell^0\ \vec{\ell}^{\,2} d|\vec{\ell}|\ d\Omega_{\vec{u}} \\ &= \frac{|\vec{\ell}|}{2}\, d\ell^0\ d\vec{\ell}^{\,2}\ d\Omega_{\vec{u}} \\ &= \frac{|\vec{\ell}|}{2}\, d\ell^0\ d\ell^2\ d\Omega_{\vec{u}} \\ &= \frac{|\vec{\ell}|}{4K}\, d(\ell-K)^2\ d\ell^2\ d\Omega_{\vec{u}} \end{aligned} \tag{4.12}$$

Now we need to decide how to choose our variables. We let

$$d\Omega_{\vec{u}} = 4\pi\,\frac{d\cos\theta}{2}\frac{d\phi}{2\pi} \tag{4.13}$$

and choose $\cos\theta/2$ with a uniform density on $(-1/2, 1/2)$ and choose $\phi/(2\pi)$ uniformly on $(0,1)$. For ℓ^2, we set

$$\ell^2 = \frac{K^2}{4}\ s_1\, r_1^{1/A} \tag{4.14}$$

where $s_1 = \pm 1$, chosen at random, and $0 < r_1 < 1$ chosen with a uniform distribution. The default value for A is $A = 1/2$. Note that this range for r_1 gives

$$-\frac{K^2}{4} < \ell^2 < \frac{K^2}{4} \ , \tag{4.15}$$

which is the range that we wanted. Then

$$d\ell^2 = \frac{2|\ell^2|}{A}\left(\frac{4|\ell^2|}{K^2}\right)^{-A}\ \frac{1}{2}\sum_{s_1} dr_1\ \ . \tag{4.16}$$

For $(\ell - K)^2$, we take the same definitions,

$$(\ell - K)^2 = \frac{K^2}{4}\ s_2\, r_2^{1/A}\ \ , \tag{4.17}$$

Then

$$d(\ell - K)^2 = \frac{2|(\ell - K)^2|}{A}\left(\frac{4|(\ell - K)^2|}{K^2}\right)^{-A}\ \frac{1}{2}\sum_{s_2} dr_2\ \ . \tag{4.18}$$

We thus have

$$\begin{aligned}
&\int d^4\ell\ \theta(|\ell^2| < K^2/4)\,\theta(|(\ell-K)^2| < K^2/4)\cdots\\
&\quad = \int \frac{|\vec{\ell}|}{4K}\, d(\ell-K)^2\ d\ell^2\ d\Omega_{\vec{u}}\ \theta(|\ell^2| < K^2/4)\,\theta(|(\ell-K)^2| < K^2/4)\cdots\\
&\quad = \frac{1}{4}\sum_{s_1,s_2}\int_0^1 dr_2\int_0^1 dr_1\int_{-1}^1 \frac{d\cos\theta}{2}\int_0^{2\pi}\frac{d\phi}{2\pi}\\
&\qquad\quad \times 4\pi\,\frac{|\vec{\ell}|}{K}\,\frac{|\ell^2|\,|(\ell-K)^2|}{A^2}\left(\frac{16|\ell^2||(\ell-K)^2|}{K^4}\right)^{-A}\cdots
\end{aligned} \tag{4.19}$$

Thus the density of points is

$$\rho = \frac{1}{4\pi}\ \frac{K}{|\vec{\ell}|}\ \frac{A^2}{|\ell^2|\,|(\ell-K)^2|}\left(\frac{16|\ell^2||(\ell-K)^2|}{K^4}\right)^{A}\ \theta(|\ell^2| < K^2/4)\,\theta(|(\ell-K)^2| < K^2/4)\ \ . \tag{4.20}$$

Note that $|\vec{\ell}|$ here can be written covariantly as

$$|\vec{\ell}| = \left[\frac{(\ell\cdot K)^2}{K^2} - \ell^2\right]^{1/2} \tag{4.21}$$

5 Two cones, lightlike separated

Consider two cones $C_+(Q_i)\cup C_-(Q_i)$ and $C_+(Q_j)\cup C_-(Q_j)$, where

$$(Q_j - Q_i)^2 = 0\ \ ,\quad Q_j^0 - Q_i^0 > 0\ \ . \tag{5.1}$$

We would like to concentrate points near these two cones, with points especially concentrated near their intersection. Let

$$K = Q_j - Q_i \tag{5.2}$$

so

$$K^2 = 0 \ . \tag{5.3}$$

Choose a reference frame in which $(K^+, K^-, \boldsymbol{K}_T) = (K^+, 0, \boldsymbol{0})$ with $K^+ > 0$. Let

$$\ell = l - Q_i \ . \tag{5.4}$$

Then the two cones are $\ell^2 = 0$ and $(\ell - K)^2 = 0$.

We use four variables, ℓ^2, $(\ell - K)^2$, the momentum fraction $x = \ell^+/K^+$ and the azimuthal angle, ϕ of $\boldsymbol{\ell}_T$. Our notation is that $\boldsymbol{\ell}_T^2 > 0$. We define x so that

$$\ell^+ = xK^+ \ . \tag{5.5}$$

We note that

$$2\ell \cdot K = \ell^2 - (\ell - K)^2 \ . \tag{5.6}$$

That is

$$\ell^- = \frac{\ell^2 - (\ell - K)^2}{2K^+} \ . \tag{5.7}$$

We also note that

$$2\ell^+\ell^- - \boldsymbol{\ell}_T^2 = \ell^2 \ , \tag{5.8}$$

where ℓ^+ is given in Eq. (5.5) and ℓ^- is given in Eq. (5.7). Thus

$$\boldsymbol{\ell}_T^2 = -(1-x)\ell^2 - x(\ell - K)^2 \ . \tag{5.9}$$

We then define

$$(\ell^1, \ell^2) = \sqrt{\boldsymbol{\ell}_T^2}\ (\cos\phi, \sin\phi) \ . \tag{5.10}$$

This gives us all four components of ℓ.

We note that because $\boldsymbol{\ell}_T^2 > 0$, the range of $\{\ell^2, (\ell - K)^2\}$ is restricted to

$$(1-x)\ell^2 + x(\ell - K)^2 < 0 \ . \tag{5.11}$$

To change variables to our new set, we write,

$$\begin{aligned}
d^4\ell &= d\ell^+\ d\ell^-\ \frac{1}{2}\, d\boldsymbol{\ell}_T^2\ d\phi \\
&= d\ell^+\ d\ell^-\ \frac{1}{2}\, d\ell^2\ d\phi \\
&= K^+dx\ \frac{d(\ell - K)^2}{2K^+}\frac{1}{2}\, d\ell^2\ d\phi \\
&= \frac{1}{4}\ d\ell^2\ d(\ell - K)^2\ dx\ d\phi \ .
\end{aligned} \tag{5.12}$$

Now we need to decide how to choose our variables. We let

$$d\phi = 2\pi\ \frac{d\phi}{2\pi} \tag{5.13}$$

and choose $\phi/(2\pi)$ uniformly on $(0,1)$. For x, we choose to cover the range

$$-\frac{1}{2} < x < \frac{3}{2} \quad . \tag{5.14}$$

We choose one of the two ranges $-1/2 < x < 1/2$ and $1/2 < x < 3/2$ at random, each with probability 1/2. For this purpose, we choose a variable j_x equal to 1 or 2 with equal probabilities. For the range $-1/2 < x < 1/2$ we choose a random number r_x uniformly in the range $0 < r_x < 1$ and set

$$x = \frac{1}{2}\, S(2r_x - 1)\, |2r_x - 1|^{1/(1-A)} \quad , \tag{5.15}$$

where A is a parameter that might be chosen as 1/2 and $S(\xi)$ denotes the sign of ξ, $S(\xi) = \xi/|\xi|$. This maps $0 < r_x < 1$ into $-1/2 < x < 1/2$. Then

$$r_x = \frac{1}{2}\left[|2x|^{1-A} S(x) + 1\right] \quad , \tag{5.16}$$

Its derivative is

$$\frac{dr_x}{dx} = \frac{1-A}{|2x|^{A}} \quad . \tag{5.17}$$

For the range $1/2 < x < 3/2$ we choose a random number r_x uniformly in the range $0 < r_x < 1$ and set

$$1 - x = \frac{1}{2}\, S(2r_x - 1)\, |2r_x - 1|^{1/(1-A)} \quad , \tag{5.18}$$

where A is a parameter that might be chosen as 1/2 and $S(\xi)$ denotes the sign of ξ, $S(\xi) = \xi/|\xi|$. This maps $0 < r_x < 1$ into $3/2 > x > 1/2$. Then

$$r_x = \frac{1}{2}\left[|2(1-x)|^{1-A} S(1-x) + 1\right] \quad , \tag{5.19}$$

Its derivative is

$$\frac{dr_x}{dx} = \frac{1-A}{|2(1-x)|^{A}} \quad . \tag{5.20}$$

We can now choose ℓ^2 and $(\ell - K)^2$. We can choose r_1 and r_2 uniformly on $(0,1)$ and set

$$\begin{aligned} \ell^2 &= M^2\, \frac{S(2r_1 - 1)}{|2r_1 - 1|^{-1/B} - 1} \quad , \\ (\ell - K)^2 &= M^2\, \frac{S(2r_2 - 1)}{|2r_2 - 1|^{-1/B} - 1} \quad . \end{aligned} \tag{5.21}$$

Here B is a parameter that could be 1/2, M^2 us a parameter with the dimension of mass squared that should be on the order of virtuality scales in the problem, and $S(\xi)$ is the sign function already introduced. This gives

$$\begin{aligned} r_1 &= \frac{1}{2}\left(1+\frac{M^2}{|\ell^2|}\right)^{-B} S(\ell^2)+\frac{1}{2}\ \ , \\ r_2 &= \frac{1}{2}\left(1+\frac{M^2}{|(\ell-K)^2|}\right)^{-B} S((\ell-K))+\frac{1}{2}\ \ . \end{aligned} \tag{5.22}$$

The jacobians are

$$\begin{aligned} \frac{dr_1}{d\ell^2} &= \frac{B}{2}\ \left(1+\frac{M^2}{|\ell^2|}\right)^{-B-1}\frac{M^2}{|\ell^2|^2}\ \ , \\ \frac{dr_2}{d(\ell-K)^2} &= \frac{B}{2}\ \left(1+\frac{M^2}{|(\ell-K)^2|}\right)^{-B-1}\frac{M^2}{|(\ell-K)^2|^2}\ \ . \end{aligned} \tag{5.23}$$

These relations map $0 < r_1 < 1$ into $-\infty < \ell^2 < \infty$ and similarly for r_2 and $(\ell - K)^2$. We need to restrict our points to $(1-x)\ell^2 + x(\ell-K)^2 < 0$. Therefore, we reject points with $(1-x)\ell^2 + x(\ell-K)^2 > 0$. When a point is rejected, we simply choose a new point. From the symmetry of the selection, we see that this rejects half the points at each step. That is, $2\theta((1-x)\ell^2 + x(\ell-K)^2 < 0)$ integrates to 1. Thus the density of points chosen is

$$\rho_{12} = 2\,\frac{B^2}{4}\ \left(1+\frac{M^2}{|\ell^2|}\right)^{-B-1}\frac{M^2}{|\ell^2|^2}\ \left(1+\frac{M^2}{|(\ell-K)^2|}\right)^{-B-1}\frac{M^2}{|(\ell-K)^2|^2}\ \ . \tag{5.24}$$

Near the light cones, the density of points is singular,

$$\rho_{12} \propto \frac{1}{|\ell^2(\ell-K)^2|^{1-B}} \tag{5.25}$$

Far from the light cones, the density falls off as

$$\rho_{12} \propto \frac{1}{|\ell^2(\ell-K)^2|^2} \tag{5.26}$$

We thus have

$$\begin{aligned} &\int d^4\ell\ \theta(-1/2<x<3/2)\cdots \\ &\quad = \frac{1}{4}\int d\ell^2\ d(\ell-K)^2\ dx\ d\phi\ \theta(-1/2<x<3/2) \\ &\qquad\quad \times\theta((1-x)\ell^2+x(\ell-K)^2<0)\cdots \\ &\quad = \int_0^1 dr_1\int_0^1 dr_2\ 2\theta((1-x)\ell^2+x(\ell-K)^2<0)\,\frac{1}{2}\sum_{j_x}\int_0^1 dr_x\ \int_0^{2\pi}\frac{d\phi}{2\pi} \\ &\qquad \times\frac{\pi}{2}\left[\frac{2\delta_{j_x,1}\ \theta\left(-\frac{1}{2}<x<\frac{1}{2}\right)|2x|^A}{1-A}+\frac{2\delta_{j_x,2}\ \theta\left(\frac{1}{2}<x<\frac{3}{2}\right)|2(1-x)|^A}{1-A}\right] \\ &\qquad \times\frac{2}{B^2}\left(1+\frac{M^2}{|\ell^2|}\right)^{B+1}\frac{|\ell^2|^2}{M^2}\left(1+\frac{M^2}{|(\ell-K)^2|}\right)^{B+1}\frac{|(\ell-K)^2|^2}{M^2}\ \cdots\ \ . \end{aligned} \tag{5.27}$$

Thus the density of points is

$$\begin{aligned}\rho = \frac{B^2(1-A)}{2\pi}&\left[\frac{\theta\left(-\frac{1}{2} < x < \frac{1}{2}\right)}{|2x|^A} + \frac{\theta\left(\frac{1}{2} < x < \frac{3}{2}\right)}{|2(1-x)|^A}\right] \\ &\times \left(1 + \frac{M^2}{|\ell^2|}\right)^{-B-1} \frac{M^2}{|\ell^2|^2} \left(1 + \frac{M^2}{|(\ell - K)^2|}\right)^{-B-1} \frac{M^2}{|(\ell - K)^2|^2} \quad .\end{aligned} \tag{5.28}$$

6 Two cones, spacelike separated

Consider two cones $C_+(Q_i) \cup C_-(Q_i)$ and $C_-(Q_j) \cup C_+(Q_j)$, where

$$(Q_j - Q_i)^2 < 0 \quad . \tag{6.1}$$

We would like to concentrate points near these two cones, with points especially concentrated near their intersection. Let

$$K = Q_j - Q_i \tag{6.2}$$

and choose a reference frame in which $K^0 = K^1 = K^2 = 0$ with $K^3 > 0$. Let

$$\ell = l - Q_i \quad . \tag{6.3}$$

Then the two cones are $\ell^2 = 0$ and $(\ell - K)^2 = 0$.

We use four variables, ℓ^2, $(\ell - K)^2$, a boost angle ω and an azimuthal angle ϕ. We note that

$$-2\ell \cdot K = (\ell - K)^2 - \ell^2 - K^2 \quad . \tag{6.4}$$

That is

$$\ell^3 = \frac{(\ell - K)^2 - \ell^2 + |K^2|}{2|K^2|^{1/2}} \quad . \tag{6.5}$$

Let us denote the part of ℓ excluding the 3-component by $\tilde{l}$. The square of these components is

$$\tilde{\ell}^2 = (\ell^0)^2 - \boldsymbol{\ell}_T^2 \quad , \tag{6.6}$$

(Here, $\boldsymbol{\ell}_T^2 \equiv (\ell^1)^2 + (\ell^2)^2 > 0$.) We then have

$$\tilde{\ell}^2 = \ell^2 + (\ell^3)^2 \tag{6.7}$$

where ℓ^3 is given in Eq. (6.5), so that

$$\tilde{\ell}^2 = \frac{1}{4|K^2|}\left\{[\ell^2 - (l - K)^2]^2 + 2|K^2|[\ell^2 + (l - K)^2] + |K^2|^2\right\} \quad . \tag{6.8}$$

We will restrict the range of ℓ^2 and $(\ell - K)^2$ to

$$\begin{aligned}|\ell^2| &< |K^2|/4 \quad , \\ |(\ell - K)^2| &< |K^2|/4 \quad .\end{aligned} \tag{6.9}$$

Then

$$\tilde{\ell}^2 > 0 \quad . \tag{6.10}$$

We can choose points satisfying Eq. (6.6) by writing

$$\begin{aligned} \ell^0 &= s_0\,[\tilde{\ell}^2]^{1/2}\,\cosh\omega \\ \ell^1 &= [\tilde{\ell}^2]^{1/2}\,\sinh\omega\,\cos\phi \\ \ell^2 &= [\tilde{\ell}^2]^{1/2}\,\sinh\omega\,\sin\phi \quad . \end{aligned} \tag{6.11}$$

Here $s_0 = \pm 1$, $0 < \omega < \infty$ and $-\pi < \phi < \pi$. (We could also use $-\infty < \omega < \infty$ and $0 < \phi < \pi$.) This gives us all four components of ℓ.

We can write the integration as

$$\begin{aligned} d^4\ell &= d\ell^0\,d|\ell_T|\,|\ell_T|d\phi\,d\ell^3 \\ &= \frac{1}{2}\,d\tilde{\ell}^2\,d\omega\,|\ell_T|d\phi\,d\ell^3 \\ &= \frac{1}{2}\,d\ell^2\,d\omega\,|\ell_T|d\phi\,d\ell^3 \\ &= \frac{1}{2}\,d\ell^2\,d\omega\,|\ell_T|d\phi\,\frac{d(\ell-K)^2}{2|K^2|^{1/2}} \\ &= \frac{1}{4|K^2|^{1/2}}\,d\ell^2\,d\omega\,[\tilde{\ell}^2]^{1/2}\sinh\omega\,d\phi\,d(\ell-K)^2 \\ &= \frac{1}{4}\left(\frac{\tilde{\ell}^2}{|K^2|}\right)^{1/2} d\ell^2\,d(\ell-K)^2\,d\cosh\omega\,d\phi \end{aligned} \tag{6.12}$$

We are ready to choose our variables. We choose ϕ uniformly on $-\pi < \phi < \pi$. We set

$$\cosh\omega = \tan\left(\frac{\pi}{4}\,(1 + r_\omega)\right) \tag{6.13}$$

and chose r_ω uniformly on $0 < r_\omega < 1$. Then

$$d\cosh\omega = \frac{\pi}{4}\,(1 + \cosh^2\omega)\,dr_\omega \tag{6.14}$$

We choose the sign s_0 of ℓ^0 in Eq. (6.11) at random, with probability 1/2 for each choice.

For ℓ^2, we set

$$\ell^2 = \frac{|K^2|}{4}\,s_1\,r_1^{1/A} \tag{6.15}$$

where $s_1 = \pm 1$, chosen at random, and $0 < r_1 < 1$ chosen with a uniform distribution. The default value for A is $A = 1/2$. Note that this range for r_1 gives

$$-\frac{K^2}{4} < \ell^2 < \frac{K^2}{4} \quad , \tag{6.16}$$

which is the range that we wanted. Then

$$d\ell^2 = \frac{2|\ell^2|}{A}\left(\frac{4|\ell^2|}{|K^2|}\right)^{-A}\ \frac{1}{2}\sum_{s_1} dr_1 \quad . \tag{6.17}$$

For $(\ell - K)^2$, we take the same definitions,

$$(\ell - K)^2 = \frac{|K^2|}{4}\ s_2\, r_2^{1/A} \quad , \tag{6.18}$$

Then

$$d(\ell - K)^2 = \frac{2|(\ell - K)^2|}{A}\left(\frac{4|(\ell - K)^2|}{|K^2|}\right)^{-A}\ \frac{1}{2}\sum_{s_2} dr_2 \quad . \tag{6.19}$$

We thus have

$$\begin{aligned}
&\int d^4\ell\ \theta(|\ell^2| < |K^2|/4)\,\theta(|(\ell - K)^2| < |K^2|/4)\cdots \\
&\quad = \int_{-|K^2|/4}^{|K^2|/4} d\ell^2 \int_{-|K^2|/4}^{|K^2|/4} d(\ell - K)^2\, \frac{1}{2}\sum_{s_0}\int_0^\infty d\cosh\omega \int_{-\pi}^{\pi}\frac{d\phi}{2\pi}\ \pi\left(\frac{\tilde\ell^2}{|K^2|}\right)^{1/2} \\
&\quad = \frac{1}{2}\sum_{s_1}\int_0^1 dr_1\ \frac{1}{2}\sum_{s_2}\int_0^1 dr_2\ \frac{1}{2}\sum_{s_0}\int_0^1 dr_\omega\ \int_{-\pi}^{\pi}\frac{d\phi}{2\pi} \\
&\quad\quad \times \frac{2|\ell^2|}{A}\left(\frac{4|\ell^2|}{|K^2|}\right)^{-A}\ \frac{2|(\ell - K)^2|}{A}\left(\frac{4|(\ell - K)^2|}{|K^2|}\right)^{-A}\ \frac{\pi}{4}\,(1 + \cosh^2\omega)\,\pi\left(\frac{\tilde\ell^2}{|K^2|}\right)^{1/2} \\
&\quad = \frac{1}{2}\sum_{s_1}\int_0^1 dr_1\ \frac{1}{2}\sum_{s_2}\int_0^1 dr_2\ \frac{1}{2}\sum_{s_0}\int_0^1 dr_\omega\ \int_{-\pi}^{\pi}\frac{d\phi}{2\pi} \\
&\quad\quad \times \frac{\pi^2}{A^2}\ |\ell^2(\ell - K)^2|\left(\frac{16|\ell^2(\ell - K)^2|}{|K^2|^2}\right)^{-A}\ \frac{\tilde\ell^2 + (\ell\cdot n)^2}{[\tilde\ell^2|K^2|]^{1/2}} \quad .
\end{aligned} \tag{6.20}$$

Here we have replaced

$$\cosh\omega^2 = \frac{(\ell^0)^2}{\tilde\ell^2} = \frac{(\ell\cdot n)^2}{\tilde\ell^2} \tag{6.21}$$

where n is a unit vector in the 0-direction in the frame that we are using. In another frame, it is obtained from $n = (1, \vec{0})$ by applying the same Lorentz transformation that turns K into the original K. One can write $\tilde\ell^2$ covariantly as

$$\tilde{l}^2 = l^2 - (l\cdot K)^2/K^2 \quad . \tag{6.22}$$

Thus the density of points is

$$\begin{aligned}
\rho = \frac{A^2}{\pi^2|\ell^2(\ell - K)^2|}\ &\left(\frac{16|\ell^2(\ell - K)^2|}{|K^2|^2}\right)^{A}\ \frac{[\tilde{l}^2\,|K^2|]^{1/2}}{\tilde\ell^2 + (\ell\cdot n)^2} \\
\times\,\theta&\left(|\ell^2| < |K^2|/4\right)\,\theta\left(|(\ell - K)^2| < |K^2|/4\right) \quad .
\end{aligned} \tag{6.23}$$

7 Two cones, lightlike separated, plus third cone

Consider two cones $C_+(Q_i) \cup C_-(Q_i)$ and $C_+(Q_j) \cup C_-(Q_j)$. Define

$$K = Q_j - Q_i \quad . \tag{7.1}$$

We suppose that Q_j and Q_i are lightlike separated:

$$K^2 = 0 \quad . \tag{7.2}$$

Then there is a pinch singularity along the line from Q_i to Q_j, that is along

$$l = (1-x)Q_i + xQ_j = Q_i + xK \tag{7.3}$$

for $0 \le x \le 1$. Suppose that there is a third cone $C_+(Q_k) \cup C_-(Q_k)$ that intersects this line. Define

$$L = Q_k - Q_i \quad . \tag{7.4}$$

Then also

$$L - K = Q_k - Q_j \quad . \tag{7.5}$$

We suppose that $L \cdot K \ne 0$ and that either

$$L^2 \ge 0 \ , \quad (L-K)^2 \le 0 \quad . \tag{7.6}$$

or else

$$L^2 \le 0 \ , \quad (L-K)^2 \ge 0 \quad . \tag{7.7}$$

Since for $l = (1-x)Q_i + xQ_j$ we have

$$(l - Q_k)^2 = (xK - L)^2 = L^2 - 2xK \cdot L \quad , \tag{7.8}$$

the intersection of all three cones is at $x = a$, where

$$a = \frac{L^2}{2K \cdot L} \quad . \tag{7.9}$$

The conditions (7.6) or (7.7) imply

$$0 \le a \le 1 \quad . \tag{7.10}$$

We would like to concentrate points along the line from Q_i to Q_j and its extension near where this line intersects the third cone. We have particularly in mind the cases $i = 1$, $j = N$ with $k = A$, $i = A+1$, $j = A$, with $k = N$, and $i = N$, $j = 1$ with $k = A+1$, $i = A$, $j = A+1$, with $k = 1$. These triple intersection regions are responsible for the double parton scattering singularity when the external momenta are near the double parton scattering configuration.

Let

$$\ell = l - Q_i \quad . \tag{7.11}$$

Then the cones $C_+(Q_i) \cup C_-(Q_i)$ and $C_+(Q_j) \cup C_-(Q_j)$ are

$$\ell^2 = 0\,, \quad (\ell - K)^2 = 0 \quad , \tag{7.12}$$

while the third cone is

$$(\ell - L)^2 = 0 \quad . \tag{7.13}$$

Choose a reference frame in which

$$\begin{aligned} (K^+, K^-, \boldsymbol{K}_T) &= (K^+, 0, \mathbf{0}) \quad , \\ (L^+, L^-, \boldsymbol{L}_T) &= (L^+, L^-, \mathbf{0}) \quad . \end{aligned} \tag{7.14}$$

Let

$$\ell = (xK^+, \ell^-, \boldsymbol{\ell}_T) \quad . \tag{7.15}$$

The pinched collinear singularity is along the line $\ell^- = 0$, $\boldsymbol{\ell}_T = 0$, with $0 \le x \le 1$. The intersection with the third cone is at $x = a$. Define also a parameter $\tilde{a}$, which should be positive, fairly small, but not too small. We will arrange to put points into the range

$$a - \tilde{a} < x < a + \tilde{a} \quad . \tag{7.16}$$

We use four variables, ℓ^2, $(\ell - K)^2$, the momentum fraction $x = \ell^+/K^+$ and the azimuthal angle, ϕ of $\boldsymbol{\ell}_T$. Our notation is that $\boldsymbol{\ell}_T^2 > 0$. We have defined x so that

$$\ell^+ = xK^+ \quad . \tag{7.17}$$

We note that

$$2\ell \cdot K = \ell^2 - (\ell - K)^2 \quad . \tag{7.18}$$

That is

$$\ell^- = \frac{\ell^2 - (\ell - K)^2}{2K^+} \quad . \tag{7.19}$$

We also note that

$$2\ell^+\ell^- - \boldsymbol{\ell}_T^2 = \ell^2 \quad , \tag{7.20}$$

where ℓ^+ is given in Eq. (7.17) and ℓ^- is given in Eq. (7.19). Thus

$$\boldsymbol{\ell}_T^2 = -(1 - x)\ell^2 - x(\ell - K)^2 \quad . \tag{7.21}$$

We then define

$$(\ell^1, \ell^2) = \sqrt{\boldsymbol{\ell}_T^2}\,(\cos\phi, \sin\phi) \quad . \tag{7.22}$$

This gives us all four components of ℓ.

We note that because $\boldsymbol{\ell}_T^2 > 0$, the range of $\{\ell^2, (\ell - K)^2\}$ is restricted to

$$(1-x)\ell^2 + x(\ell - K)^2 < 0 \ . \tag{7.23}$$

To change variables to our new set, we write,

$$\begin{aligned} d^4\ell &= d\ell^+ \ d\ell^- \ \frac{1}{2}\, d\boldsymbol{\ell}_T^2 \ d\phi \\ &= d\ell^+ \ d\ell^- \ \frac{1}{2}\, d\ell^2 \ d\phi \\ &= K^+ dx \ \frac{d(\ell-K)^2}{2K^+}\frac{1}{2}\, d\ell^2 \ d\phi \\ &= \frac{1}{4}\ d\ell^2 \ d(\ell-K)^2 \ dx \ d\phi \ . \end{aligned} \tag{7.24}$$

Now we need to decide how to choose our variables. We let

$$d\phi = 2\pi \ \frac{d\phi}{2\pi} \tag{7.25}$$

and choose $\phi/(2\pi)$ uniformly on $(0,1)$. For x, we choose to cover the range

$$-\tilde{a} < x - a < \tilde{a} \ . \tag{7.26}$$

We choose a random number r_x uniformly in the range $0 < r_x < 1$ and set

$$x - a = \tilde{a}\, S(2r_x - 1)\, |2r_x - 1|^{1/A} \ , \tag{7.27}$$

where A is a parameter that might be chosen as 1/2 and $S(\xi)$ denotes the sign of ξ, $S(\xi) = \xi/|\xi|$. This maps $0 < r_x < 1$ into $-\tilde{a} < x - 1 < \tilde{a}$. Then

$$r_x = \frac{1}{2}\left[\left|\frac{x-a}{\tilde{a}}\right|^{A} S(x-a) + 1\right] \ , \tag{7.28}$$

Its derivative is

$$\frac{dr_x}{dx} = \frac{A}{2\tilde{a}}\ \left|\frac{x-a}{\tilde{a}}\right|^{A-1} \ . \tag{7.29}$$

We can now choose ℓ^2 and $(\ell - K)^2$. We can choose r_1 and r_2 uniformly on $(0,1)$ and set

$$\begin{aligned} \ell^2 &= M^2 \ \frac{S(2r_1 - 1)}{|2r_1 - 1|^{-1/B} - 1} \ , \\ (\ell - K)^2 &= M^2 \ \frac{S(2r_2 - 1)}{|2r_2 - 1|^{-1/B} - 1} \ . \end{aligned} \tag{7.30}$$

Here B is a parameter that could be $1/2$, M^2 us a parameter with the dimension of mass squared that should be on the order of virtuality scales in the problem, and $S(\xi)$ is the sign function already introduced. This gives

$$\begin{aligned} r_1 &= \frac{1}{2}\left(1+\frac{M^2}{|\ell^2|}\right)^{-B} S(\ell^2) + \frac{1}{2}\ , \\ r_2 &= \frac{1}{2}\left(1+\frac{M^2}{|(\ell-K)^2|}\right)^{-B} S((\ell-K)) + \frac{1}{2}\ . \end{aligned} \tag{7.31}$$

The jacobians are

$$\begin{aligned} \frac{dr_1}{d\ell^2} &= \frac{B}{2}\ \left(1+\frac{M^2}{|\ell^2|}\right)^{-B-1} \frac{M^2}{|\ell^2|^2}\ , \\ \frac{dr_2}{d(\ell-K)^2} &= \frac{B}{2}\ \left(1+\frac{M^2}{|(\ell-K)^2|}\right)^{-B-1} \frac{M^2}{|(\ell-K)^2|^2}\ . \end{aligned} \tag{7.32}$$

These relations map $0 < r_1 < 1$ into $-\infty < \ell^2 < \infty$ and similarly for r_2 and $(\ell - K)^2$. We need to restrict our points to $(1-x)\ell^2 + x(\ell-K)^2 < 0$. Therefore, we reject points with $(1-x)\ell^2 + x(\ell-K)^2 > 0$. From the symmetry of the selection, we see that this rejects half the points at each step. When a point is rejected, we could simply choose a new point. Somewhat more efficiently, we can choose the mirror image point obtained by reversing the signs of l^2 and $(l-K)^2$. That is, $2\theta((1-x)\ell^2 + x(\ell-K)^2 < 0)$ integrates to 1. Thus the density of points chosen is

$$\rho_{12} = 2\,\frac{B^2}{4}\ \left(1+\frac{M^2}{|\ell^2|}\right)^{-B-1} \frac{M^2}{|\ell^2|^2}\ \left(1+\frac{M^2}{|(\ell-K)^2|}\right)^{-B-1} \frac{M^2}{|(\ell-K)^2|^2}\ . \tag{7.33}$$

Near the light cones, the density of points is singular,

$$\rho_{12} \propto \frac{1}{|\ell^2(\ell-K)^2|^{1-B}} \tag{7.34}$$

Far from the light cones, the density falls off as

$$\rho_{12} \propto \frac{1}{|\ell^2(\ell-K)^2|^2} \tag{7.35}$$

We thus have

$$
\begin{aligned}
&\int d^4\ell\ \theta(a-\tilde{a}<x<a+\tilde{a})\cdots \\
&\quad =\frac{1}{4}\int d\ell^2\ d(\ell-K)^2\ dx\ d\phi\ \theta(a-\tilde{a}<x<a+\tilde{a}) \\
&\qquad\quad \times\theta((1-x)\ell^2+x(\ell-K)^2<0)\cdots \\
&\quad =\int_0^1 dr_1\int_0^1 dr_2\ 2\theta((1-x)\ell^2+x(\ell-K)^2<0)\int_0^1 dr_x\ \int_0^{2\pi}\frac{d\phi}{2\pi} \\
&\qquad \times\frac{\pi}{2}\ \frac{2\tilde{a}}{A}\ \left|\frac{x-a}{\tilde{a}}\right|^{1-A} \\
&\qquad \times\frac{2}{B^2}\left(1+\frac{M^2}{|\ell^2|}\right)^{B+1}\frac{|\ell^2|^2}{M^2}\left(1+\frac{M^2}{|(\ell-K)^2|}\right)^{B+1}\frac{|(\ell-K)^2|^2}{M^2}\ \cdots\ .
\end{aligned}
\tag{7.36}
$$

Thus the density of points is

$$
\begin{aligned}
\rho=&\frac{AB^2}{2\pi\tilde{a}}\left|\frac{\tilde{a}}{x-a}\right|^{1-A}\left(1+\frac{M^2}{|\ell^2|}\right)^{-B-1}\frac{M^2}{|\ell^2|^2}\left(1+\frac{M^2}{|(\ell-K)^2|}\right)^{-B-1}\frac{M^2}{|(\ell-K)^2|^2} \\
&\times\theta(a-\tilde{a}<x<a+\tilde{a})\ .
\end{aligned}
\tag{7.37}
$$

Note that x can be expressed covariantly as

$$
x=\frac{\ell\cdot L-a\ \ell\cdot K}{K\cdot L}\ . \tag{7.38}
$$

8 Two cones, lightlike separated, plus third cone, modified

There is an alternate way to do this. We can choose l^2 and $(l-K)^2$ as described above, leaving the choice of x for the end and omitting the rejection of l^2 and $(l-K)^2$ points based on the sign of $(1-x)\ell^2+x(\ell-K)^2$.

We consider $(\ell-L)^2$, using $L^+/K^+=L^2/(2K\cdot L)=a$,

$$
\begin{aligned}
(\ell-L)^2 &=\ell^2-2\ell\cdot L+L^2 \\
&=\ell^2-2\ell^+L^--2\ell^-L^++L^2 \\
&=\ell^2-2xK^+L^--(\ell^2-(\ell-K)^2)\frac{L^+}{K^+}+L^2 \\
&=\ell^2-x\,2K\cdot L-(\ell^2-(\ell-K)^2)\,a+a\,2K\cdot L \\
&=(1-a)\ell^2-a(\ell-K)^2+a\,2K\cdot L-x\,2K\cdot L\ .
\end{aligned}
\tag{8.1}
$$

We will use $(\ell-L)^2$ as our last independent variable instead of x. We have

$$
dx=\frac{d(\ell-L)^2}{|2K\cdot L|}\ . \tag{8.2}
$$

We will need to write x in terms of $(\ell - L)^2$,

$$x = \frac{(1-a)\ell^2 - a(\ell - K)^2 + a\,2K\cdot L}{2K\cdot L} - \frac{(\ell - L)^2}{2K\cdot L} \quad . \tag{8.3}$$

From Eq. (7.21), we have

$$\boldsymbol{\ell}_T^2 = -\ell^2 + x[\ell^2 - (\ell - K)^2] \quad . \tag{8.4}$$

Thus

$$\begin{aligned} \boldsymbol{\ell}_T^2 = & -\ell^2 + \frac{(1-a)\ell^2 - a(\ell - K)^2 + a\,2K\cdot L}{2K\cdot L}[\ell^2 - (\ell - K)^2] \\ & - \frac{\ell^2 - (\ell - K)^2}{2K\cdot L}\,(\ell - L)^2 \quad . \end{aligned} \tag{8.5}$$

This gives

$$(\ell - L)^2 = \lambda_0^2 - \frac{2K\cdot L}{\ell^2 - (\ell - K)^2}\,\boldsymbol{\ell}_T^2 \quad . \tag{8.6}$$

where

$$\lambda_0 = -\frac{\ell^2\,2K\cdot L}{\ell^2 - (\ell - K)^2} + (1-a)\ell^2 - a(\ell - K)^2 + a\,2K\cdot L \quad . \tag{8.7}$$

This implies

$$\begin{aligned} (\ell - L)^2 < \lambda_0 \qquad & \text{if } \frac{2K\cdot L}{\ell^2 - (\ell - K)^2} > 0 \quad , \\ (\ell - L)^2 > \lambda_0 \qquad & \text{if } \frac{2K\cdot L}{\ell^2 - (\ell - K)^2} < 0 \quad . \end{aligned} \tag{8.8}$$

We now discuss the integration over $(l - L)^2$. Let us call $(l - L)^2 = \lambda$. Suppose that $2K\cdot L/[\ell^2 - (\ell - K)^2] > 0$, so that we need to implement an integration

$$\int_{\lambda_0}^{\infty} d\lambda\ \cdots \quad . \tag{8.9}$$

Define

$$w(\lambda) = \frac{1}{2}\left(1 + \frac{M^2}{|\lambda|}\right)^{-B} S(\lambda) \tag{8.10}$$

where $S(\lambda)$ is the sign of λ, that is $\lambda/|\lambda|$. We note that

$$\begin{aligned} w(\infty) = & \ \frac{1}{2} \quad , \\ w(-\infty) = & \ -\frac{1}{2} \quad , \end{aligned} \tag{8.11}$$

and

$$dw = \frac{B}{2}\left(1+\frac{M^2}{|\lambda|}\right)^{-B-1}\frac{M^2}{\lambda^2}\,d\lambda\ \ . \tag{8.12}$$

Thus

$$\int_{\lambda_0}^{\infty} d\lambda\ \cdots = \int_{w(\lambda_0)}^{1/2} dw\ \left[\frac{B}{2}\left(1+\frac{M^2}{|\lambda|}\right)^{-B-1}\frac{M^2}{\lambda^2}\right]^{-1}\cdots\ \ . \tag{8.13}$$

We can make a further change of variables to

$$r = \frac{w(\lambda)-w(\lambda_0)}{1/2-w(\lambda_0)} \tag{8.14}$$

Then r ranges from 0 to 1. We have

$$\int_{\lambda_0}^{\infty} d\lambda\ \cdots = \int_0^1 dr\ \frac{1}{\rho(\lambda)}\cdots\ \ . \tag{8.15}$$

where

$$\rho(\lambda) = \frac{1}{1/2-w(\lambda_0)}\ \frac{B}{2}\left(1+\frac{M^2}{|\lambda|}\right)^{-B-1}\frac{M^2}{\lambda^2}\ \ . \tag{8.16}$$

Similar formulas would apply for the case $2K\cdot L/[\ell^2-(\ell-K)^2]<0$, for which we want to integrate over λ from $-\infty$ to λ_0.

The only problem with this scheme is that, depending on the values of l^2 and $(l-K)^2$, the integration over λ may not include $\lambda = 0$. Then we can have a high density of points in a region in which we do not need them.