The examination papers are numbered in the upper right-hand corner of each page. Print and then sign your name in the spaces provided on this page. For identification purposes, be sure to submit this page together with your answers when the exam is finished. Be sure to place both the exam number and the question number on any additional pages you wish to have graded.

You are encouraged to use the constants and other information on the following two pages where appropriate to help you solve the problems.

There are six equally weighted questions, each beginning on a new page. Read all six questions before attempting any answers.

Begin each answer on the same page as the question, but continue on additional blank pages if necessary. Write only on one side of each page. Each page should contain work related to only one problem. When you start a new problem, start a new page.

If you need to leave your seat, wait until everyone else is seated before approaching the proctor.

Calculators may be used only for arithmetic and will be provided. **Personal calculators are not allowed.** Dictionaries may be used if they have been approved by the proctor before the examination begins. **Electronic dictionaries are not allowed. No other papers or books may be used.**

When you have finished, come to the front of the room. For each problem, put the pages in order and staple them together. Then put all problems in numerical order and place them in the envelope provided. Finally, hand the envelope to the proctor.

Please make sure you follow all instructions carefully. If you fail to follow instructions, or to hand in your exam paper on time, an appropriate number of points may be subtracted from your final score.
Constants

- Electron charge \( (e) \) \( 1.60 \times 10^{-19} \text{ C} \)
- Electron rest mass \( (m_e) \) \( 9.11 \times 10^{-31} \text{ kg (0.511 MeV/c}^2) \)
- Proton rest mass \( (m_p) \) \( 1.673 \times 10^{-27} \text{ kg (938 MeV/c}^2) \)
- Neutron rest mass \( (m_n) \) \( 1.675 \times 10^{-27} \text{ kg (940 MeV/c}^2) \)
- Atomic mass unit (AMU) \( 1.7 \times 10^{-27} \text{ kg} \)
- Atomic weight of a hydrogen atom \( 1 \text{ AMU} \)
- Atomic weight of a nitrogen atom \( 14 \text{ AMU} \)
- Atomic weight of an oxygen atom \( 16 \text{ AMU} \)
- Planck's constant \( (h) \) \( 6.63 \times 10^{-34} \text{ J} \cdot \text{s} \)
- Speed of light in vacuum \( (c) \) \( 3.00 \times 10^8 \text{ m/s} \)
- Boltzmann's constant \( (k_B) \) \( 1.38 \times 10^{-23} \text{ J/K} \)
- Gravitational constant \( (G) \) \( 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2 \)
- Permeability of free space \( (\mu_0) \) \( 4\pi \times 10^{-7} \text{ H/m} \)
- Permittivity of free space \( (\varepsilon_0) \) \( 8.85 \times 10^{-12} \text{ F/m} \)
- Mass of earth \( (M_E) \) \( 5.98 \times 10^{24} \text{ kg} \)
- Equatorial radius of earth \( (R_E) \) \( 6.38 \times 10^6 \text{ m} \)
- Mass of sun \( (M_S) \) \( 1.99 \times 10^{30} \text{ kg} \)
- Radius of sun \( (R_S) \) \( 6.96 \times 10^8 \text{ m} \)
- Classical electron radius \( (r_0) \) \( 2.82 \times 10^{-15} \text{ m} \)
- Density of water \( 1.0 \text{ kg/liter} \)
- Density of ice \( 0.917 \text{ kg/liter} \)
- Specific heat of water \( 4180 \text{ J/(kg K)} \)
- Specific heat of ice \( 2050 \text{ J/(kg K)} \)
- Heat of fusion of water \( 334 \text{ kJ/kg} \)
- Heat of vaporization of water \( 2260 \text{ kJ/kg} \)
- Specific heat of oxygen \( (c_V) \) \( 21.1 \text{ J/mole-K} \)
- Specific heat of oxygen \( (c_P) \) \( 29.4 \text{ J/mole-K} \)
- Gravitational acceleration on Earth \( (g) \) \( 9.8 \text{ m/s}^2 \)
- 1 atmosphere \( 1.01 \times 10^5 \text{ Pa} \)

Pauli spin matrices

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Trigonometric identities

\[
1 - \cos \theta = 2 \sin^2(\theta/2)
\]
\[
1 + \cos \theta = 2 \cos^2(\theta/2)
\]
\[
\sin \theta = 2 \sin(\theta/2) \cos(\theta/2)
\]

Stirling's formula

\[
\log(N!) \approx N \log N - N
\]
Problem 1

A particle of mass $m$ sits in an infinite one-dimensional square well potential of length $L$ extending from $x = 0$ to $x = L$. At $t = 0$, the particle is described by the wave function:

$$\Psi(x, 0) = \frac{1}{\sqrt{L}} \left[ \sin \left( \frac{\pi x}{L} \right) + \sin \left( \frac{2\pi x}{L} \right) \right]$$

(a) What is the probability to find the particle in the ground state at $t = 0$?
(b) What is the wavefunction as a function of time, $t$?
(c) What is the probability to find the particle between $x = 0$ and $x = L/2$ at time $t$?
Problem 2

The Hamiltonian for the spin of an electron in a magnetic field $\mathbf{B}$ is

$$\hat{H} = -\frac{e}{mc} \hat{\mathbf{S}} \cdot \mathbf{B}$$

with $e$ and $m$ the electron charge and mass respectively and $c$ the speed of light. $\hat{\mathbf{S}} = \frac{\hbar}{2} \sigma$ is the spin operator with $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ the Pauli matrices.

(a) Show that the Heisenberg equation of motion for $\hat{\mathbf{S}}$ is

$$\frac{d}{dt} \hat{\mathbf{S}}(t) = \frac{\omega}{B} \hat{\mathbf{S}}(t) \times \mathbf{B}$$

with $\omega = eB/mc$ the cyclotron frequency.

(b) Solve the equation of motion for $\hat{\mathbf{S}}(t)$, given $\mathbf{B} = (0, 0, B)$ with the initial conditions $\hat{\mathbf{S}}(t = 0) = (\hat{S}^0_x, \hat{S}^0_y, \hat{S}^0_z)$.

(c) Let $\mathbf{B} = 0$, and let the electron spin be in an eigenstate of $\hat{S}_z$ with eigenvalue $\hbar/2$. At time $t = 0$, a field $\mathbf{B} = (0, 0, B)$ is suddenly switched on. What state is the spin in at time $t = T > 0$?
Problem 3

Consider a one-dimensional, quantum harmonic oscillator of mass $m$ and frequency $\omega$, in thermal equilibrium with temperature $T$. Compute $\langle x^2 \rangle$ for this distribution. It may help to recall that

$$x = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger).$$
Problem 4

Consider a system containing a large number of independent, non-interacting molecules. Each molecule is shaped like an equilateral triangle, with spins located at each corner. The spin projection of the \( i \)-th atom along the \( z \) axis, which is perpendicular to the plane of the triangles, may adopt only the two values \( S_i = \pm 1 \). The atoms interact (1) with an external magnetic field through their magnetic dipole moment of magnitude \( \mu \), and (2) with each other through an exchange interaction \( J \), \( U_{ij} = -JS_iS_j \).

\[
\begin{array}{c}
\text{H} \\
\text{H} \\
\text{H}
\end{array}
\]

(a) Write an expression for the total energy of the spins in one such molecule, in a magnetic field \( \mathbf{H} \).

(b) Find the partition function for the spin system of a single molecule in field \( \mathbf{H} \).

(c) Find the free energy associated with the spin system of a single molecule at temperature \( T \) and field \( \mathbf{H} \).

(d) Find an expression for the average spin \( \langle S \rangle \) on a single molecule at temperature \( T \) and field \( \mathbf{H} \).

(e) Comment on the average value of \( \langle S \rangle \) at low temperature \( T \ll |J|/k_B \), as a function of the magnetic field.
Problem 5

(a) Define the thermodynamic potential *Enthalpy*.

(b) In a Joule-Thomson expansion, a gas is expanded from high pressure ($p_1$) to a low pressure ($p_2 < p_1$) by passing it through a porous plug. The plug is assumed to transfer no heat to or from the gas passing through it. The figure illustrates the porous plug in the center of a thermally insulated tube with pistons on either side. The force on each piston is kept constant to maintain $p_1$ and $p_2$ at their initial values throughout the process while $V_1$ shrinks and $V_2$ expands.

Show that the Enthalpy is conserved in a Joule-Thomson expansion.

(c) For the Joule-Thomson process in (b), show that the temperature of an ideal gas does not change.
Problem 6

This question asks you to consider whether a carton of store-bought milk contains an equilibrium colloidal suspension.

Consider a container of milk, with height \( h = 0.25 \) m. We can idealize the milk as a suspension of fat droplets of diameter \( r = 0.5 \) \( \mu \)m in water. The mass density of water is \( \rho_{\text{water}} = 1 \) g/cm\(^3\) and that of fat is \( \rho_{\text{fat}} = 0.91 \) g/cm\(^3\). The forces acting on the droplets are gravity and buoyancy.

(a) Assuming a suspension in equilibrium, find \( c(z)/c(0) \), the concentration of fat droplets as a function of height \( z \), normalized by the concentration at height zero. You don’t have to insert numerical values in this part.

(b) What is the characteristic length scale associated with the concentration profile? Give a numerical answer.

(c) Is the result in (b) consistent with the assumption that typical store-bought milk is an equilibrium colloidal suspension?